Coming Home for Blocking Transitions Fast

Christopher T. Schwanen[1], Wied Pakusa[2], and Wil M. P. van der Aalst[1]

[1] Chair of Process and Data Science (PADS),
RWTH Aachen University, Aachen, Germany
{schwanen,wvdaalst}@pads.rwth-aachen.de

[2] Faculty of Mathematics, Informatics and Technology,
Koblenz University of Applied Sciences, Koblenz, Germany
pakusa@hs-koblenz.de

Abstract Gaujal, Haar and Mairesse proved in 2003 that every cluster in a live, bounded, free-choice Petri net system has a blocking marking, i.e., a marking which enables only the transitions of the cluster. Moreover, for each cluster, this blocking marking is unique and reachable from every state of the system. In this paper, we study live, bounded, free-choice systems that, in addition, possess a home cluster, i.e., a cluster whose blocking marking only marks the places of the cluster. Prototypical examples are sound free-choice workflow nets, the standard model class in process mining. Our main result shows that in such systems, shortest firing sequences to blocking markings have linear length. Moreover, we obtain a quadratic upper bound on the length of shortest firing sequences between arbitrary markings. Both results strongly improve known bounds of Desel and Esparza for live, bounded, free-choice systems in presence of a home cluster. Furthermore, we show that each transition can be enabled via an elementary firing sequence, i.e., a firing sequence which contains at most one transition from every cluster. As an important technical contribution, we show that CP-subnets are acyclic or intersect the home cluster. These insights might be of independent interest for future research on free-choice systems.

Keywords: Petri Nets · Free-Choice Petri Nets · Home Cluster · Blocking Marking · Reachability · Process Mining

1 Introduction

We study live, bounded, free-choice Petri net systems (LBFC-systems). Recall that a Petri net system is *live*, if at each state, every transition can be enabled eventually; it is *bounded* if there is a global constant $b \in \mathbb{N}$ such that no place contains more than b tokens in any reachable marking; and it is *free-choice* if all pairs of transitions either share all or none of their input places.

LBFC-systems enjoy several useful algorithmic and structural properties. For instance, they allow for a combinatorial, graph-theoretic characterization of their recurrent states. To be precise, Best et al. [7] proved that *home markings* (i.e., markings to which the system can return from every reachable state) in LBFC-systems can be characterized by so-called *traps*, which are sets of places that, once marked, cannot get unmarked. Since maximal unmarked traps can be computed via a polynomial-time fixed-point process, this gives rise to efficient (polynomial-time) algorithms to analyze LBFC-systems. As one application, it becomes tractable to decide if an LBFC-system is cyclic (i.e., whether the initial marking is a home marking). Note that such behavioral properties of systems are usually very hard to compute, so it is quite beneficial to have such efficient algorithms at hand (see, e.g., [15, 16]). Another remarkable property of LBFC-systems is that the diameter of their reachability graph is polynomially-bounded in the size of the system and the place-bound b. In fact, as Desel and Esparza [12, 13] established in their *Shortest Sequence Theorem* whenever two markings are reachable from each other, then there exists a connecting firing sequence with length $\mathcal{O}(b \cdot n^3)$ where n is the number of transitions and b the place-bound of the LBFC-system. A third notable result on LBFC-systems is due to Gaujal et al. [19] and is known as the *Blocking Theorem*. It states that for every *cluster* in an LBFC-system (i.e., sets of transitions with the same input places), there exists a unique *blocking marking* that enables only the transitions of the cluster. Moreover, this blocking marking is reachable from every reachable marking of the system *and* without firing any transition of the cluster itself. Uniqueness implies that whenever a marking enables only a single cluster, it coincides with the blocking marking of that cluster already. This means that the state space of an LBFC-system, when restricted to blocking markings, is in a one-to-one correspondence with the clusters of the system (in particular, it is linearly bounded in the number of transitions). Of course this raises many questions, for example, of how this "simple" part of the state space is connected to the remainder of the system or how we can reach this part as fast as possible. We study these questions in the present paper.

Given such outstanding properties of LBFC-systems, it is not surprising that they found numerous applications also in related areas of computer science, such as process mining [6, 9]. Here, algorithmic problems involving large amounts of event data have to be solved efficiently. Process models, which are often based on Petri nets, play a central role in this context. In particular, *sound workflow nets*, a prominent subclass of *live* and *bounded* Petri net systems, tailored for business process modeling, are nowadays the de-facto standard for models in the process mining community [2–5]. Since sound workflow nets are live and bounded already, by adding the free-choice restriction, we obtain a subclass of LBFC-systems to which we can apply the host of structure-theoretical and algorithmic results such as the ones outlined above. This route has been investigated by several researchers in the process mining community, see, e.g., [1, 18, 20, 21], and has lead to important algorithmic improvements.

While this shows how Petri net theory helps to solve problems in different application domains, the converse is true as well: notions arising, e.g., in the area of process mining, can lead to new insights on the underlying Petri net model. A prominent example is the concept of *home clusters*, which originated from the work on the workflow net model in the process mining community. A home cluster in a free-choice Petri net is a cluster whose (unique) blocking marking only marks the places of the cluster itself. This means that the transitions in a home cluster bind *all* tokens such that no tokens are left outside the cluster. The consequence is that a home cluster captures the idea of having a “regeneration point” of the system which can serve as a “terminal/initial state”, i.e., a state where the system completes its computation or restarts afresh. In fact, in *sound* workflow nets, one requirement is that the model has the *option to complete* which corresponds to the existence of a home cluster (where the net reaches its final marking). While home clusters clearly have practical motivation, they also have very interesting structure-theoretical properties. As the third author proved in a series of articles [2–5], the effect of having a home cluster alone in combination with the free-choice assumption is sufficient in order to guarantee *lucency* of the system. Lucency is a generalization of the blocking marking property saying that each (reachable) marking is uniquely determined by the *set of enabled transitions*. In other words, it is impossible to have two different reachable markings that enable the same set of transitions. An immediate consequence is that the state space of a free-choice system with a home cluster is finite, i.e., the system is bounded.

By combining these threads together, we see that sound *free-choice* workflow nets are LBFC-systems with a home cluster, and the presence of the home cluster has strong structural implications. Another consequence of liveness and the home cluster property is, for example, that sound workflow nets are *cyclic*, i.e., that the initial marking is a home marking that can be visited again and again. For cyclic LBFC-systems, in turn, Desel and Esparza [11] proved that their reachability sets consist of all those markings which agree on all S-invariants and mark each proper trap (both properties which can be checked efficiently). This makes the fundamental problem of reachability for sound free-choice workflow nets solvable efficiently.

Motivated by these results, we continue the study of LBFC-systems with a home cluster (which includes sound free-choice workflow nets). We focus on the question of how efficiently blocking markings can be reached in such systems, i.e., we ask for the *length of shortest firing sequences to blocking markings*—that is, sequences which cannot be shortened by permuting and/or removing transitions. Our motivation comes twofold: first, blocking markings are a fundamental concept in the theory of free-choice systems, and understanding how efficiently they can be reached is of theoretical interest. Second, in algorithmic contexts such as process mining, it is often necessary to construct firing sequences to blocking markings (e.g., when replaying event logs on process models), and having good upper bounds on the length of such sequences is important for performance guarantees. Of course, there can be shortest firing sequences of different lengths, so another question is: what is the *minimal length* of a shortest firing sequence to a blocking marking? We already have a cubic upper bound due to Desel and Esparza’s Shortest Sequence Theorem [12, 13], but one could hope that this bound can be improved by the presence of a home cluster.

2 Preliminaries

In this paper, $\mathbb{N} \coloneqq \{0, 1, 2, \ldots\}$ denotes the natural numbers (including 0). Let us fix our notation for the basic Petri net concepts used throughout the paper.

Definition 1 (Indicator Function). Let A be a set and let $B \subseteq A$. The function $\chi_B \colon A \to \{0, 1\}$, defined by

$$\chi_B(x) \coloneqq \begin{cases} 1 & x \in B, \\ 0 & x \notin B, \end{cases}$$

is the *indicator function* (or *characteristic function*) of set B. As a special case, we have, $\chi_\emptyset \eqqcolon \mathbf{0}$ where $\mathbf{0}$ is the zero function mapping every element of A to 0.

Definition 2 (Multiset). A *multiset* M over a set A is a function $M \colon A \to \mathbb{N}$; thus, for any $a \in A$, $M(a)$ indicates the multiplicity of a in the multiset M. The set of all multisets over A is given by $\mathbb{N}^A$. We also use the notation $\left[a^{M(a)} \mid a \in A\right]$ for a multiset $M \in \mathbb{N}^A$. Any set $B \subseteq A$ can be considered a multiset using its indicator function, i.e., $\chi_B = \left[b^1 \mid b \in B\right] = [b \in B] \in \mathbb{N}^A$. For multisets $M, M' \in \mathbb{N}^A$, we use the standard notation for functions, e.g., $M + M'$, $M \leq M'$, etc. The *support* (or *domain*) of a multiset $M \in \mathbb{N}^A$, denoted by $\langle M \rangle$, is the set of elements contained in M, i.e., $\langle M \rangle \coloneqq \{a \in A \mid M(a) > 0\}$. The restriction of a multiset $M \in \mathbb{N}^A$ to a set $B \subseteq A$ is the multiset $M|_B \coloneqq \left[a^{M(a)} \mid a \in B\right] \in \mathbb{N}^B$ containing only the elements of B.

Whenever we compare or operate with multisets $M \in \mathbb{N}^A$ and $M' \in \mathbb{N}^B$ over different domains A and B, we implicitly assume that they are first extended with 0 for each item of $A \cup B$ not in their domain.

Definition 3 (Sequence, Parikh Vector, Permutation). *Sequences* with index set I over a set A are denoted by $\sigma = \langle a_i \rangle_{i \in I} \in A^I$. The *length* of a sequence σ is written as $|\sigma|$ and the set of all finite sequences over A is denoted by A^*. Given two sequences σ and σ', $\sigma \cdot \sigma'$ (or $\sigma\sigma'$ in short) denotes the concatenation of the two sequences. The restriction of a sequence $\sigma \in A^*$ to a set $B \subseteq A$ is the subsequence $\sigma|_B$ of σ consisting of all elements in B. The *Parikh vector* of a sequence $\sigma \in A^*$, denoted by $\boldsymbol{\sigma}$, is its multiset representation defined by $\boldsymbol{\sigma}(a) \coloneqq \left|\sigma|_{\{a\}}\right|$ for every $a \in A$, and $\langle \boldsymbol{\sigma} \rangle$ provides the *support* of σ. A sequence $\sigma' \in A^*$ is a *permutation* of σ if and only if $\boldsymbol{\sigma'} = \boldsymbol{\sigma}$.

Definition 4 (Petri Net). A *Petri net* N is a bipartite directed graph $N = (P, T, F)$ where P and T, $P \cap T = \emptyset$ are disjoint finite sets of vertices and $F \subseteq (P \times T) \cup (T \times P)$ is the set of arcs. In a Petri net, P is called the set of *places*, T the set of *transitions*, and F the *flow relation*. Given a vertex $v \in P \cup T$, its *pre-set* $\bullet v$ and *post-set* $v\bullet$ are defined by $\bullet v \coloneqq \{u \in P \cup T \mid (u, v) \in F\}$ and $v\bullet \coloneqq \{u \in P \cup T \mid (v, u) \in F\}$. With regard to a place (transition), its pre- and post-set are also called *input* and *output transitions* (*places*). This $\bullet$-notation is extended to a set of vertices $V \subseteq P \cup T$ by $\bullet V \coloneqq \bigcup_{v \in V} \bullet v$ and $V\bullet \coloneqq \bigcup_{v \in V} v\bullet$.

Definition 5 (Incidence Matrix of a Petri Net). Let $N = (P, T, F)$ be a Petri net. Its *incidence matrix* $\mathbf{N}\colon (P \times T) \to \{-1, 0, +1\}$ is defined by

$$\mathbf{N}(p,t) \coloneqq \begin{cases} -1 & \text{if } (p,t) \in F \wedge (t,p) \notin F, \\ +1 & \text{if } (p,t) \notin F \wedge (t,p) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6 ((Induced) Subnet). Let $N = (P, T, F)$ be a Petri net. $N' = (P', T', F')$ is a *subnet* of N if $P' \subseteq P$, $T' \subseteq T$, and $F' = F \cap ((P' \times T') \cup (T' \times P'))$. Note that a subnet is always an *induced* subnet (since F' contains all arcs of F between the nodes in $P' \cup T'$). For a set of nodes $V \subseteq P \cup T$, we denote this *(induced) subnet* as $N[V]$ given as $N[V] \coloneqq (P \cap V, T \cap V, F \cap (V \times V))$.

Definition 7 (Free-Choice Petri Net). A Petri net $N = (P, T, F)$ is *free-choice* if any two transitions either share all or none of their input places, i.e., $\forall t, t' \in T\colon \bullet t = \bullet t' \vee \bullet t \cap \bullet t' = \emptyset$.

Definition 8 (Marking, System, Firing Rule). Given a Petri net $N = (P, T, F)$, a *marking* $M \in \mathbb{N}^P$ is a multiset where $M(p)$ is the number of *tokens* at place $p \in P$. A place $p \in P$ is *marked* at M if $M(p) > 0$. The pair (N, M) of a Petri net $N = (P, T, F)$ and a marking $M \in \mathbb{N}^P$ is called a *system*. A transition $t \in T$ is *enabled* in M, denoted by $(N, M)[t\rangle$, if and only if each of its input places $p \in \bullet t$ is marked, i.e., $M \geq \chi_{\bullet t}$. An enabled transition may *fire*, denoted by $(N, M)[t\rangle(N, M')$, and firing results in a new marking $M' \coloneqq M - \chi_{\bullet t} + \chi_{t \bullet}$.

A sequence of transitions $\sigma = \langle t_i \rangle_{i=1}^{n} \in T^*$ is called a *firing sequence* of (N, M) if for every transition t_i of the sequence holds that $(N, M_{i-1})[t_i\rangle$ and $(N, M_{i-1})[t_i\rangle(N, M_i)$ where $M_0 = M$ and $M_n = M'$. Firing such a sequence is denoted by $(N, M)[\sigma\rangle(N, M')$. The empty sequence $\langle\rangle$ is always enabled and firing the empty sequence leaves the marking unchanged, i.e., $(N, M)[\langle\rangle\rangle(N, M)$. A marking M' is *reachable* if a firing sequence $\sigma \in T^*$ exists such that M' is the resulting marking, i.e., $(N, M)[\sigma\rangle(N, M')$. The set of all reachable markings of (N, M) is defined by $[N, M\rangle \coloneqq \left\{ M' \in \mathbb{N}^P \;\middle|\; \exists \sigma \in T^*\colon (N, M)[\sigma\rangle(N, M') \right\}$.

Definition 9 (Boundedness, Safeness). Given some $k \in \mathbb{N}$, a system (N, M_0) with $N = (P, T, F)$ is *k-bounded* if k is a bound for any reachable marking, i.e., $\forall M \in [N, M_0\rangle\colon \forall p \in P\colon M(p) \leq k$. (N, M_0) is *safe* if it is 1-bounded.

Definition 10 ((Quasi-)Liveness). Let (N, M_0) with $N = (P, T, F)$ be a system. A transition $t \in T$ is *quasi-live* if there exists a reachable marking $M' \in [N, M_0\rangle$ which enables t. A transition $t \in T$ is *live* if it is quasi-live for every reachable marking $M \in [N, M_0\rangle$, i.e., $\forall M \in [N, M_0\rangle\colon \exists M' \in [N, M\rangle\colon (N, M')[t\rangle$. A system (N, M_0) is *(quasi-)live* if every transition $t \in T$ is (quasi-)live.

Definition 11 (Well-Formedness). A Petri net N is *well-formed* if there exists a marking M_0 of N such that (N, M_0) is a live and bounded system.

Definition 12 (LBFC-System). We denote systems (N, M_0) which are live, (b-)bounded and free-choice systems as *LBFC-system*. Note that the place bound b

may vary across different LBFC-systems, but is, of course, fixed for each individual LBFC-system.

Definition 13 (Home Marking, Cyclic System). Let (N, M_0) be a system. A marking $M \in [N, M_0\rangle$ is a *home marking* if it can be reached from any reachable marking, i.e., $\forall M' \in [N, M_0\rangle \colon M \in [N, M'\rangle$. A system (N, M_0) is *cyclic* if its initial marking M_0 is a home marking.

Definition 14 (Cluster). Let $N = (P, T, F)$ be a Petri net and let $v \in P \cup T$. The *cluster* of v, denoted by $[v]_C$, is the minimal set of places and transitions that includes

- v, i.e., $v \in [v]_C$,
- all output transitions of contained places, i.e., $\forall p \in [v]_C \cap P \colon p\bullet \in [v]_C$, and
- all input places of contained transitions, i.e., $\forall t \in [v]_C \cap T \colon \bullet t \in [v]_C$.

The set of all clusters of N is defined by $[N]_C \coloneqq \{[v]_C \mid v \in P \cup T\}$.

It is not hard to show that the set of clusters forms a partition of the set $P \cup T$ (see [12, Proposition 4.5]). What is more, in free-choice Petri nets, clusters have a very simple structure: they consist of a set of places and a set of transitions such that all places are connected to all transitions as input places and vice versa (i.e., they form a complete bipartite subgraph). Also the cluster of an element $x \in P \cup T$ can be determined quite easily: $[x]_C = \bullet x \cup (\bullet x)\bullet$ if x is a transition, and $[x]_C = x\bullet \cup \bullet(x\bullet)$ if x is a place. As a consequence, in free-choice Petri nets, if one transition of a cluster is enabled in some marking, then all transitions of the cluster are enabled in that marking (see [12, Proposition 4.6]).

3 Shortest Enabling Sequences in T-Systems

As a warmup, we start our analysis of shortest sequences with what we call *enabling sequences*, i.e., sequences that *enable* a given cluster from some reachable marking (and have minimal length with this property). The motivation is as follows: if we want to reach a blocking marking of some cluster, there are two steps involved. First, we need to *enable* the cluster (if it is not enabled already), and second, we need to *disable* all other clusters different from the target cluster. Thus, understanding how to enable clusters efficiently is an important building block for reaching blocking markings. Moreover, we start our investigation with *live T-systems* (instead of considering full LBFC-systems). This first step is justified by the fact that T-systems are, in a precise sense, the basic *building blocks* of LBFC-systems as shown by Desel and Esparza [12, Chapter 7] building on earlier results due to Esparza [14] and Esparza and Suárez [17]. Let us start with the two basic definitions of enabling sequences and T-systems.

Definition 15 (Enabling Sequence). Let (N, M_0) with $N = (P, T, F)$ be a system. For a transition $t \in T$ and some reachable marking $M \in [N, M_0\rangle$, a firing sequence $\sigma \in T^*$ enabled at M is a *t-enabling sequence (from M)* if it results in a marking that enables t, i.e., $(N, M)[\sigma\langle t\rangle\rangle$. A t-enabling sequence $\eta \in T^*$ is a *shortest t-enabling sequence* if no strict (permuted) subsequence $\sigma \in T^*$, $\boldsymbol{\sigma} < \boldsymbol{\eta}$, is t-enabling as well.

In other words, a shortest t-enabling sequence enables t and cannot be shortened by removing transitions or permuting them. Of course, shortest t-enabling sequences never contain t itself:

Proposition 1. *Let (N, M_0) with $N = (P, T, F)$ be a system. Let $t \in T$ be some transition and let $\sigma \in T^*$ be a shortest sequence that enables t, i.e., $(N, M_0)[\sigma\langle t\rangle\rangle$. Then, $t \notin \langle\boldsymbol{\sigma}\rangle$.*

A T-net is a Petri net where each place has exactly one input and one output transition (in particular, T-nets are free-choice).

Definition 16 (T-Net, T-System). A Petri net (P, T, F) is a *T-net* if every place has exactly one input and one output transition, i.e., $\forall p \in P\colon |\bullet p| = 1 = |p\bullet|$. A system (N, M) is a *T-system* if N is a T-net.

T-systems cannot express choices (conflicts) between transitions. As a matter of definition, no pair of transitions $t, t' \in T, t \neq t'$ can compete for tokens from the same input places (formally: $\bullet t \cap \bullet t' = \emptyset$). In general, firing sequences in which all pairs of transitions satisfy this property are known as *biased firing sequences* and have been studied by Desel and Esparza [12, 13]. We recall some of their results and apply them to shortest enabling sequences in live T-systems.

Definition 17 (Biased Firing Sequence). Let (N, M_0) with $N = (P, T, F)$ be a system. A firing sequence $\sigma \in T^*$ is *biased* if $\bullet t \cap \bullet t' = \emptyset$ for every two distinct transitions $t, t' \in \langle\boldsymbol{\sigma}\rangle, t \neq t'$ that occur in σ.

Lemma 1 ([12, Lemma 3.23, 13, Lemma 3.2]). *Let (N, M_0) with $N = (P, T, F)$ be a system. Let $t \in T$ be some transition and let $\sigma_1\sigma_2\langle t\rangle$ be a biased firing sequence such that $\sigma_1, \sigma_2 \in (T \setminus \{t\})^*$ be two firing sequences with $\langle\boldsymbol{\sigma_1}\rangle \supseteq \langle\boldsymbol{\sigma_2}\rangle$. If $(N, M_0)[\sigma_1\sigma_2\langle t\rangle\rangle(N, M)$, then also $(N, M_0)[\sigma_1\langle t\rangle\sigma_2\rangle(N, M)$.*

Lemma 2 ([12, Lemma 3.24, 13, Lemma 3.3]). *Let (N, M_0) with $N = (P, T, F)$ be a system and let $\sigma \in T^*$ be a biased firing sequence. Then, there exists a permutation σ' of σ with $\boldsymbol{\sigma'} = \boldsymbol{\sigma}$ such that $\sigma' = \sigma_1\sigma_2$ with $\boldsymbol{\sigma_1} \leq \chi_T$ and $\langle\boldsymbol{\sigma_1}\rangle \supseteq \langle\boldsymbol{\sigma_2}\rangle$.*

There is another way to rearrange biased firing sequences: every first occurrence of a transition that is already enabled in the current marking can be moved forward.

Lemma 3. *Let (N, M_0) with $N = (P, T, F)$ be a system. Let $t, t' \in T, t \neq t'$ be two distinct transitions and let $\langle t\rangle\sigma\langle t'\rangle$ be a biased firing sequence such that $\sigma \in (T \setminus \{t, t'\})^*$ and $(N, M_0)[\langle t\rangle\sigma\langle t'\rangle\rangle(N, M)$. If t' is enabled in M_0, i.e., $(N, M_0)[t'\rangle$, then also $(N, M_0)[\langle t', t\rangle\sigma\rangle(N, M)$.*

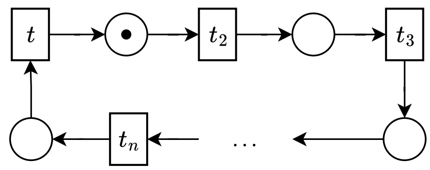

Figure 1. A live T-system for which the bound in Theorem 1 is tight.

Proof. Since $(N, M_0)[t'\rangle$, $M_0 \geq \chi_{\bullet t'}$ and because $\langle t\rangle\sigma\langle t'\rangle$ is biased, neither t nor any transition occurring in σ has one of the places $\bullet t'$ as input place, i.e., $\bullet t' \cap (\bullet t \cup \bullet\langle\boldsymbol{\sigma}\rangle) = \emptyset$. Thus, their occurrences do not disable t' and vice versa. Hence, $(N, M_0)[\langle t', t\rangle\sigma\rangle(N, M)$. □

With Lemmas 1 and 2 we can already show how to shorten biased enabling sequences such that we obtain an enabling sequence which contains each transition at most once.

Lemma 4 (Biased Enabling Sequence Lemma). *Let (N, M_0) with $N = (P, T, F)$ be a system. Let $t \in T$ be some transition and let $\sigma \in T^*$ be a biased sequence that enables t, i.e., $(N, M_0)[\sigma\langle t\rangle\rangle$. Then, there exists a (permuted) subsequence $\eta \in (T \setminus \{t\})^*$ with $\boldsymbol{\eta} \leq \boldsymbol{\sigma}$ such that $(N, M_0)[\eta\langle t\rangle\rangle$ and no transition occurs more than once in η, i.e., $\boldsymbol{\eta} \leq \chi_{T\setminus\{t\}}$.*

Proof. Let us assume that $t \notin \langle\boldsymbol{\sigma}\rangle$ (otherwise choose the prefix before the first occurrence of t as σ). According to Lemma 2, there exists a permutation $\sigma' \in (T \setminus \{t\})^*$ of σ with $\boldsymbol{\sigma'} = \boldsymbol{\sigma}$ such that $(N, M_0)[\sigma'\langle t\rangle\rangle$ and $\sigma' = \sigma_1\sigma_2$ with $\sigma_1 \leq \chi_T$ and $\langle\boldsymbol{\sigma_1}\rangle \supseteq \langle\boldsymbol{\sigma_2}\rangle$. Then, also $(N, M_0)[\sigma_1\langle t\rangle\sigma_2\rangle$ according to Lemma 1. Choose $\eta = \sigma_1$ and the claim follows. □

Since all firing sequences in T-systems are biased, we can apply the result above to live T-systems:

Theorem 1 (Shortest Enabling Sequence Theorem). *Let (N, M_0) with $N = (P, T, F)$ be a live T-system and let $t \in T$. Then, there exists a t-enabling sequence $\eta \in (T \setminus t)^*$ which fires each transition at most once, i.e., $\boldsymbol{\eta} \leq \chi_{T\setminus\{t\}}$.*

Proof. Since (N, M_0) is live, there exists a shortest t-enabling sequence $\eta \in T^*$. Because N is a T-net, every firing sequence in (N, M_0) is biased, including η. Hence, the claim follows by Lemma 4 (since η was chosen to be minimal). □

Note that Theorem 1 does not only prove the existence of a shortest enabling sequence in a live T-system with length of at most $|T| - 1$, but establishes the more general result that *every* shortest enabling sequence satisfies this bound.

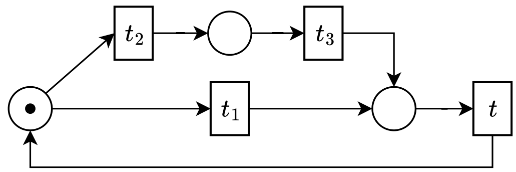

Figure 2. An LBFC-system with two different shortest t-enabling sequences $\langle t_1 \rangle$ and $\langle t_2, t_3 \rangle$ of different lengths.

Also, it is easy to see that this bound is tight: in the live T-system in Figure 1 every other transition in T must fire to enable transition t.

Actually, for the case of T-systems, we can show an even stronger result: *all* shortest t-enabling sequences are permutations of each other, i.e., their Parikh vectors are identical. This means that in a live marking of a T-system each transition t has a *canonical* enabling sequence (up to the order of transitions).

Proposition 2. *Let (N, M_0) with $N = (P, T, F)$ be a live T-system and let $\eta, \eta' \in T^*$ be minimal t-enabling sequences for some transition $t \in T$, i.e., $(N, M)[\eta\langle t \rangle\rangle$, $(N, M)[\eta'\langle t \rangle\rangle$. Then, $\boldsymbol{\eta} = \boldsymbol{\eta'}$.*

Proof. Let $\eta, \eta' \in T^*$ be two t-enabling sequences of minimal length. By Lemma 4, both sequences contain each transition at most once. We proceed by induction on $|\eta|$ and show that $\boldsymbol{\eta} = \boldsymbol{\eta'}$. If t is enabled at M_0 (i.e., $|\eta| = 0$), the claim holds trivially since both sequences are empty. Otherwise, there exists an unmarked place $p \in \bullet t$. This place has a *unique* pre-transition $t_p \in \bullet p$ since N is a T-net. Of course, this transition must occur in both sequences to mark place p. Hence, η and η' have a least one common transition t_p.

Let $\eta = \alpha_1\langle t_p \rangle\alpha_2$ and $\eta' = \beta_1\langle t_p \rangle\beta_2$. Using Lemma 3, we can rearrange $\alpha_1\langle t_p \rangle$ and $\beta_1\langle t_p \rangle$ into two sequences $\alpha_1^1\langle t_p \rangle\alpha_1^2$ and $\beta_1^1\langle t_p \rangle\beta_1^2$ such that α_1^1 and β_1^1 are two minimal t_p-enabling sequences (which can be fired from M_0). Since $|\alpha_1^1| < |\eta|$, we can apply the induction hypothesis and obtain $\boldsymbol{\alpha_1^1} = \boldsymbol{\beta_1^1}$. Hence, $\alpha_1^1\langle t_p \rangle$ and $\beta_1^1\langle t_p \rangle$ can be fired from M_0 and result in the same marking M_1. From here, the remaining sequences $\alpha_1^2\alpha_2$ and $\beta_1^2\beta_2$ are both t-enabling sequences from M_1 and $|\alpha_1^2\alpha_2| < |\eta|$. Thus, we can again apply the induction hypothesis and obtain $\boldsymbol{\eta} = \boldsymbol{\eta'}$.

□

Of course, we cannot hope for an analogous result for LBFC-systems. The reason is that choices (conflicts) can lead to completely different shortest enabling sequences (also of different lengths). A simple example is shown in Figure 2 where two incomparable shortest t-enabling sequences exist.

4 Homing Sequences and Disabling Clusters

Before we turn our attention to enabling sequences in general LBFC-systems, we study how to *disable* clusters different from the target cluster. The key insight is that it is always possible to reach the *home cluster* of an LBFC-system from any reachable marking via an *elementary homing sequence* that fires at most one transition from each cluster. Later, such homing sequences can then be utilized in order to disable all clusters different from the target cluster. Home clusters have first been studied by the third author in a line of research on *lucency* [cf. 4].

Definition 18 (Home Cluster). Let (N, M_0) with $N = (P, T, F)$ be a system. A cluster $C \in [N]_C$ is a *home cluster* of (N, M_0) if and only if the marking $\chi_{C\cap P} \in \mathbb{N}^P$ is reachable and a home marking, i.e., $\forall M \in [N, M_0\rangle \colon \chi_{C\cap P} \in [N, M\rangle$. If such a cluster C exists, we say that (N, M_0) has a home cluster.

A home cluster has strong implications on the underlying Petri net structure as we will show next. First, it is known that LBFC-systems can be covered by simple types of subnets, namely by so-called S- and T-components (see Theorems 3 and 4 below). An S-component is, in particular, a strongly-connected S-net, where an S-net is the dual concept to a T-net: each transition has exactly one input and one output place. In this way, S-nets cannot express concurrency (since transitions can neither synchronize on multiple input places nor distribute tokens to multiple output places). One immediate consequence is that the number of tokens in an S-component is invariant and, since the LBFC-system is live, each S-component must contain at least one token. On the other hand, since a home cluster can be reached from any reachable marking, it must intersect all S-components (since in the marking of a home cluster all other places are unmarked). Let us make these ideas precise.

Definition 19 (S-Net, S-System). A Petri net (P, T, F) is an *S-net* if every transition has exactly one input and one output place, i.e., $\forall t \in T \colon |\bullet t| = 1 = |t\bullet|$. A system (N, M) is an *S-system* if N is an S-net.

Definition 20 (S-Component, T-Component). Let $N = (P, T, F)$ be a Petri net. A subnet $N[P' \cup T']$ induced by the sets $P' \subseteq P$ and $T' \subseteq T$ is an *S-component* of N if it is a strongly-connected S-net and contains every input and output transition of a contained place, i.e., $\forall p \in P' \colon \bullet p \cup p\bullet \subseteq T'$. A subnet $N[P' \cup T']$ induced by the sets $P' \subseteq P$ and $T' \subseteq T$ is a *T-component* of N if it is a strongly-connected T-net and contains every input and output place of a contained transition, i.e., $\forall t \in T' \colon \bullet t \cup t\bullet \subseteq P'$.

Proposition 3 ([12, Proposition 5.2 (2)]). *Let (N, M_0) be a system and let $N_S = (P', T', F')$ be an S-component of N. Then, for any two markings $M \in [N, M_0\rangle$ and $M' \in [N, M\rangle$, $\sum_{p\in P'} M(p) = \sum_{p\in P'} M'(p)$.*

Proposition 4 ([12, Proposition 5.12 (2)]). *Let (N, M_0) be a system and let $N_T = (P', T', F')$ be a T-component of N. Then, for any sequence of transitions $\sigma \in T'^*$, $(N, M_0)[\sigma\rangle(N, M)$ if and only if $(N_T, M_0|_{P'})[\sigma\rangle(N_T, M|_{P'})$.*

Theorem 2 (Activation of T-Components [12, Theorem 5.20]**).** *Let $N_T = (P', T', F')$ be a T-component of an* LBFC*-system (N, M_0) with $N = (P, T, F)$. There exists a firing sequence $\sigma \in (T \setminus T')^*$ with $(N, M_0)[\sigma\rangle(N, M)$ such that $(N_T, M|_{P'})$ is live.*

Theorem 3 (S-Coverability Theorem [12, Theorem 5.6]**).** *Well-formed free-choice Petri nets are covered by S-components.*

Theorem 4 (T-Coverability Theorem [12, Theorem 5.18]**).** *Well-formed free-choice Petri nets are covered by T-components.*

Proposition 5. *Let (N, M_0) be an* LBFC*-system and let $C \in [N]_C$ be one of its clusters. C is a home cluster only if it intersects all S-components (and this intersection contains precisely one place from the home cluster).*

Proof. Let us assume that there is an S-component $N_S = (P', T', F')$ of $N = (P, T, F)$ that does not intersect the home cluster C, i.e., $P' \cap C = \emptyset$. That implies that the marking of the home cluster $\chi_{C \cap P}$ does not mark any places in P'. By Proposition 3, the number of tokens within an S-component is stable. Therefore, there is no possible marking $M \in \mathbb{N}^P$ such that some place in P' is marked and C is a home cluster of the system (N, M). Thus, if C is a home cluster, (N, M_0) cannot be live. This is a contradiction. Hence, C is not a home cluster if it does not intersect all S-components.

Now, of course an S-component cannot contain two different input places from the home cluster (because then, it must contain a common successor transition as well, contradicting the definition of an S-net). Also, the intersection cannot just contain a transition (since an S-component must contain *some* input place of each contained transition and this input place is part of the home cluster as well).

□

An immediate consequence is that an LBFC-system with a home cluster is safe:

Corollary 1. *Let (N, M_0) be an* LBFC*-system with a home cluster. Then, (N, M_0) is safe.*

Definition 21 (Homing Sequence). Let (N, M_0) with $N = (P, T, F)$ be a system with a home cluster $C \subseteq P \cup T$ and let $M \in [N, M_0\rangle$ be some reachable marking. A firing sequence $\sigma \in T^*$ enabled at M is a *homing sequence* if it results in the marking of the home cluster, i.e., $(N, M)[\sigma\rangle(N, \chi_{C \cap P})$.

According to the definition of a home cluster (Definition 18), some homing sequence always exists. Next, we show the much stronger result that for every marking a homing sequence can be constructed that contains at most one transition per cluster (in particular, it is of length at most $|[N]_C| - 1$). We call such sequences *elementary*:

Definition 22 (Elementary Sequence). Let (N, M_0) with $N = (P, T, F)$ be a system. A firing sequence $\sigma \in T^*$ is *elementary* if it contains at most one transition per cluster and each transition $t \in T$ occurs at most once, i.e., $\forall t \in T\colon \sum_{t \in [t]_C} \sigma(t) \leq 1$.

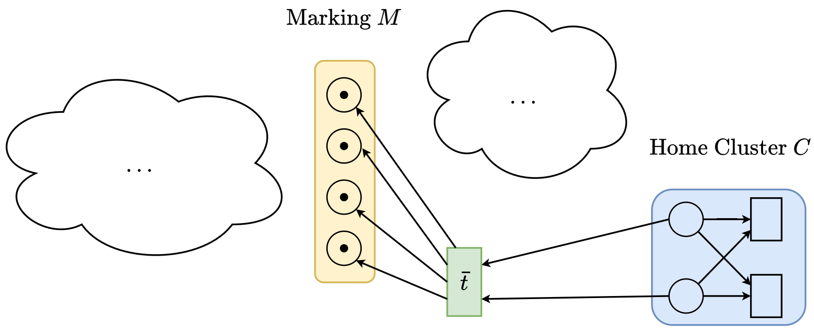

Figure 3. Extension of the LBFC-system (N, M_0) with home cluster C by a new transition $\bar{t} \notin T$ that reestablishes the marking M from the home cluster marking $\chi_{C \cap P}$.

Proposition 6. *Elementary firing sequences are biased.*

In general, when we want to reach a blocking marking of a transition t, i.e., a marking that only enables $[t]_C$, it might happen that, on our way, we encounter markings that enable t but also other clusters as well. For the home cluster, however, this is not possible as the following theorem states.

Theorem 5 ([4, Theorem 5.5])**.** *Let (N, M_0) be a free-choice system with a home cluster C. If all places of the home cluster $C \cap P$ are marked at a reachable marking $M \in [N, M_0\rangle$, then they are the only places marked at M, i.e., $\forall M \in [N, M_0\rangle \colon M \geq \chi_{C \cap P} \implies M = \chi_{C \cap P}$.*

As a last preparation, we require a result on the realizability of T-vectors in T-systems. It states that in a live T-system it is possible to construct a firing sequence that fires each transition exactly once (in some order).

Lemma 5 ([8, Theorem 5.31, 10, Theorems 6 and 7])**.** *Let (N, M) with $N = (P, T, F)$ be a T-system. (N, M) is live if and only if χ_T is realizable, i.e., $\exists \sigma \in T^* \colon (N, M)[\sigma\rangle \wedge \boldsymbol{\sigma} = \chi_T$.*

We are now ready to prove the main result of this section.

Theorem 6 (Shortest Homing Sequence Theorem). *Let (N, M_0) with $N = (P, T, F)$ be an LBFC-system with home cluster C and let $M \in [N, M_0\rangle$ be a reachable marking. Then, there exists a homing sequence $\eta \in (T \setminus C)^*$ from M such that η is elementary and thus of length at most $|[N]_C| - 1$.*

Proof. Because C is a home cluster, there exists a sequence $\eta \in (T \setminus C)^*$ enabled in M and resulting in $\chi_{C \cap P}$, i.e., $(N, M)[\eta\rangle(N, \chi_{C \cap P})$.

We can extend N with a transition $\bar{t} \notin T$ such that $\bullet\bar{t} = C \cap P$ and $\bar{t}\bullet = \langle M\rangle$ to retrieve the extended net $\bar{N} = (P, T \cup \{\bar{t}\}, F \cup \{(p, \bar{t}) \mid p \in C \cap P\} \cup \{(\bar{t}, p) \mid$

$p \in \langle M \rangle\}$) as shown in Figure 3. Because we only added a transition, $[\bar{N}, M_0\rangle \supseteq [N, M_0\rangle$ and because $\bullet\bar{t} = C \cap P$, $\bar{N}$ is still free-choice. Note that $\chi_{C\cap P}$ is a home marking of (N, M_0) and also the only marking of (N, M_0) that enables the home cluster. Since $\chi_{C\cap P} \in [\bar{N}, M_0\rangle$, $\bar{t}$ is enabled in $\chi_{C\cap P}$ and firing results in M, i.e., $(\bar{N}, \chi_{C\cap P})[\bar{t}\rangle(\bar{N}, M)$. But, because $M \in [N, M_0\rangle$, there are also no other markings in $(\bar{N}, M_0)$ enabling the cluster of $\bar{t}$. Hence, $[\bar{N}, M_0\rangle = [N, M_0\rangle$, and $(\bar{N}, M_0)$ is also live and bounded and has a home cluster $C \cup \{\bar{t}\}$.

Due to the well-formedness of $\bar{N}$ and Theorem 4, there exists some T-component $\bar{N}_T = (P', T', F')$ of $\bar{N}$ covering $\bar{t}$, i.e., $\bar{t} \in T'$. By Theorem 2, this T-component can be activated by some firing sequence $\sigma \in ((T \cup \{\bar{t}\}) \setminus T')^*$, i.e., $\exists \sigma \in ((T \cup \{\bar{t}\}) \setminus T')^*$ with $(\bar{N}, M_0)[\sigma\rangle(\bar{N}, M')$ such that $(\bar{N}_T, M'|_{P'})$ is live and safe. Note that σ is also a firing sequence enabled in (N, M_0), i.e., $(N, M_0)[\sigma\rangle$. Now, according to Proposition 4 there is also a firing sequence $\sigma' \in T'^*$ enabled in M' that results in some marking which enables $\bar{t}$, i.e., $(\bar{N}, M')[\sigma'\langle\bar{t}\rangle\rangle$. But, because of Theorem 5, $\chi_{C\cap P} = \chi_{C\cap P'}$ is the only marking enabling $\bar{t}$. Hence, $(\bar{N}_T, \chi_{C\cap P'})$ is live and safe and firing $\bar{t}$ results in $M|_{P'} = M$, i.e., $(\bar{N}_T, \chi_{C\cap P'})[\bar{t}\rangle(\bar{N}_T, M|_{P'})$.

Using Lemma 5, we can infer that $\chi_{T'\setminus\{\bar{t}\}}$ is realizable from M in $\bar{N}$ (because $\chi_{T'}$ is realizable from $\chi_{C\cap P}$) and results in the marking of the home cluster $\chi_{C\cap P}$. Because our additional transition $\bar{t}$ does not occur in $\chi_{T'\setminus\{\bar{t}\}}$, it is also realizable in N. Therefore, there exists a homing sequence $\eta \in (T \setminus C)^*$ such that $\boldsymbol{\eta} = \chi_{T'\setminus\{\bar{t}\}}$ and because T' contains at most one transition of each cluster, η is elementary and the claim follows.

□

With this preparation, we can show how clusters can be disabled by elementary firing sequences in order to reach blocking markings.

Definition 23 (Blocking Marking). Let (N, M_0) with $N = (P, T, F)$ be a system. For some transition $t \in T$, a reachable marking $M \in [N, M_0\rangle$ is a *blocking marking* associated to t if t is enabled in M, i.e., $(N, M)[t\rangle$, and every other transition either shares all of its input places with t or is not enabled in M, i.e., $\forall t' \in T\colon (N, M)[t'\rangle \implies \bullet t' \subseteq \bullet t$.

Of course, in free-choice systems, blocking markings are associated to clusters (due to the free-choice property either none or all transitions of a cluster are enabled in a marking [cf. 12, Proposition 4.6]). Remarkably, in LBFC-systems, blocking markings are unique:

Theorem 7 (Blocking Theorem [19, Theorem 3.1, 22, Theorem 2.4]**).** *Let (N, M_0) with $N = (P, T, F)$ be an LBFC-system. Then, every cluster $C \in [N]_C$ has a unique blocking marking $M \in [N, M_0\rangle$ associated to it. Furthermore, M is a home marking which can be obtained from any reachable marking without firing a transition in C, i.e., $\forall M' \in [N, M_0\rangle\colon \exists \sigma \in (T \setminus C)^*\colon (N, M')[\sigma\rangle(N, M)$.*

A much more general property can be shown in presence of a home cluster: the enabled transitions uniquely identify the current marking. This property was termed *lucency* by the third author [cf. 4].

Theorem 8 **([2, Theorem 3]).** *Let (N, M_0) with $N = (P, T, F)$ be an* LBFC-*system with a home cluster and let $M_1, M_2 \in [N, M_0\rangle$ be two reachable markings. If M_1 and M_2 enable exactly the same transitions, i.e., $\{t \in T \mid (N, M_1)[t\rangle\} = \{t \in T \mid (N, M_2)[t\rangle\}$, then $M_1 = M_2$.*

We are ready to prove the second main result of this section: in every marking of an LBFC-system with a home cluster, it is possible to disable an arbitrary set of clusters via an elementary firing sequence.

Theorem 9 **(Disabling Cluster Theorem).** *Let (N, M_0) with $N = (P, T, F)$ be an* LBFC-*system with a home cluster C and let $M \in [N, M_0\rangle$ be a reachable marking. Furthermore, let $\mathcal{C} \subseteq [N]_C$ be the set of all clusters enabled at M, i.e., $\mathcal{C} \coloneqq \{[t]_C \in [N]_C \mid (N, M)[t\rangle\}$. For any non-empty subset of clusters $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' \neq \emptyset$ there exists an elementary firing sequence $\sigma \in (T \setminus C)^*$ leading to a marking M', i.e., $(N, M)[\sigma\rangle(N, M')$, where only the clusters $\mathcal{C}'$ are enabled, i.e., $\{t \in T \mid (N, M')[t\rangle\} = \cup\mathcal{C}' \cap T$.*

Proof. First observe that for every (non-empty) $\mathcal{C}' \subseteq \mathcal{C}$ there can be at most one marking $M' \in [N, M_0\rangle$ that enables exactly the clusters in $\mathcal{C}'$. This follows directly from Theorem 8. In particular, if $\mathcal{C}' = \mathcal{C}$ we can choose $\sigma = \langle\rangle$ and the claim follows.

Let us assume that $\mathcal{C}' \subset \mathcal{C}$. First of all, note that $|\mathcal{C}| > 1$ (because otherwise we could not disable the single cluster). Therefore, $C \notin \mathcal{C}$, because whenever the home cluster is enabled, it is the only cluster enabled (see Theorem 5). Since N is free-choice, the only way to disable a cluster in $\mathcal{C} \setminus \mathcal{C}'$ is to fire one of its transitions. Let $\eta \in (T \setminus C)^*$ be an elementary homing sequence from M, which results in the marking of the home cluster $\chi_{C \cap P}$. Here, only C is enabled. In particular, no transition of any cluster in $\mathcal{C}$ is enabled in $\chi_{C \cap P}$. Therefore, at least one transition of every cluster in $\mathcal{C}$ occurs in η. In fact, it is exactly one transition of each cluster occurring exactly once in η because η is elementary. Also, there cannot be other enabled clusters in $[N]_C \setminus (\{C\} \cup \mathcal{C})$ of which no transition occurs in η because then, the tokens enabling such an additional cluster would remain in their places after firing η which contradicts Theorem 5 that after firing η only the places of C are marked.

The remainder of the proof is done by induction on the length of η.

Base case: $|\eta| = |\mathcal{C}|$. Then, every transition $t \in \langle\boldsymbol{\eta}\rangle$ must be enabled. Because firing η results in $\chi_{C \cap P}$ and every transition occurring in η is enabled, all of their output places are in the home cluster and each transition has a different set of output places, i.e., $\langle\boldsymbol{\eta}\rangle\bullet \subseteq C \cap P$ and $\forall t, t' \in \langle\boldsymbol{\eta}\rangle, t \neq t' \colon t\bullet \cap t'\bullet = \emptyset$. Now, we can use Lemma 3 to move the transitions of $\mathcal{C} \setminus \mathcal{C}'$ occurring in η forward and by firing them disabling all clusters in $\mathcal{C} \setminus \mathcal{C}'$. Without firing a transition from $\mathcal{C}'$, the home cluster cannot yet be enabled and therefore, we have reached M'.

Induction step: $|\eta| > |\mathcal{C}|$. Now, for every cluster in $\mathcal{C}$, there is exactly one transition occurring in η that is enabled. As above, we can use Lemma 3 to move the transitions of $\mathcal{C} \setminus \mathcal{C}'$ occurring in η forward and by firing them disabling all clusters in $\mathcal{C} \setminus \mathcal{C}'$. This, however, might enable additional clusters $\mathcal{C}'' \subseteq [N]_C \setminus (\{C\} \cup (\mathcal{C} \setminus \mathcal{C}'))$. In this case, take the remaining homing sequence $\eta|_{\cup(\mathcal{C}' \cup \mathcal{C}'') \cap T}$

with $|\eta|_{\cup(\mathcal{C}'\cup\mathcal{C}'')\cap T}| < |\eta|$ and proceed on $\mathcal{C}' \cup \mathcal{C}''$ to disable the clusters in $\mathcal{C}''$. If no additional clusters have been enabled, we have already reached M'.

□

Corollary 2. *Let (N, M_0) with $N = (P, T, F)$ be an* LBFC*-system with a home cluster C and let $M \in [N, M_0\rangle$ be a reachable marking enabling the transitions of some cluster $C' \in [N]_C$. Then, the blocking marking M' associated to cluster C' can be obtained by an elementary firing sequence $\eta \in (T \setminus C')^*$, i.e., $(N, M)[\eta\rangle(N, M')$.*

5 Shortest Sequences in LBFC-Systems with Home Cluster

We next establish the main result of the paper: we show that from every marking in an LBFC-system with a home cluster and for every transition t, we can reach the blocking marking associated to t via a sequence of linear length. Actually, we show something slightly stronger: *every* shortest sequence to the blocking marking is of linear length. We further show that shortest sequences between arbitrary pairs of reachable markings are of quadratic length. This improves upon the best known upper bound of cubic length by the Shortest Sequence Theorem of Desel and Esparza [12, Theorem 9.17]. Recall that LBFC-systems with home cluster are safe, cf. Corollary 1, so the bound of the Shortest Sequence Theorem yields $\mathcal{O}(n^3)$ where n denotes the number of transitions.

Our strategy is as follows. Since we already saw that we can *disable* clusters via elementary firing sequences, it would be sufficient to show that we can *enable* a single cluster via an elementary sequence as well. If we then combine the *enabling* and *disabling* sequences, we obtain a sequence which visits each cluster at most twice, i.e., a sequence of linear length. To this end, let us introduce the generalized notion of k-elementary firing sequences.

Definition 24 (k-Elementary Firing Sequence). Let (N, M_0) with $N = (P, T, F)$ be a system. A firing sequence $\sigma \in T^*$ is *k-elementary* if at most k transitions for each cluster occur in σ, i.e., $\forall t \in T\colon \sum_{t\in[t]_C} \sigma(t) \leq k$.

Note that a 1-elementary firing sequence is just an elementary firing sequence. Furthermore, let $en_C(M)$ denote the set of clusters enabled at marking M, i.e.,

$$en_C(M) \coloneqq \{C \in [N]_C \mid \exists t \in C \cap T\colon (N, M)[t\rangle\}.$$

Thus in order to establish our result, it suffices to show that *every* shortest firing sequence to a blocking marking is 2-elementary. To this end, we show that if a shortest sequence would vist some cluster more than twice, then we could rearrange it into a shorter sequence which reaches the blocking marking as well (of course, this is impossible when we started with a *shortest* sequence). Our main technical tool is the following lemma which allows us to rearrange firing sequences by *expediting* transitions (this approach resembles proof ideas used in the work on lucency [cf. 4]).

Lemma 6. *Let (N, M_0) with $N = (P, T, F)$ be a free-choice system and let $\sigma \in T^*$ be a firing sequence from M_0 to some reachable marking M, i.e., $(N, M_0)[\sigma\rangle(N, M)$, of the form:*

$$\sigma = \sigma_1 \langle t \rangle \sigma_2,$$

for some transition $t \in T$ and firing sequences $\sigma_1, \sigma_2 \in T^*$. Furthermore, let M_1 be the marking reached after firing σ_1 from M_0, i.e., $(N, M_0)[\sigma_1\rangle(N, M_1)$. Then, by expediting transitions from σ_2, we can rearrange σ into a firing sequence $\sigma' \in T^*$ with $(N, M_0)[\sigma'\rangle(N, M)$ of the form:

$$\sigma' = \sigma'_1 \langle t \rangle \sigma'_2,$$

such that:

- $\boldsymbol{\sigma'} = \boldsymbol{\sigma}$*,*
- $\boldsymbol{\sigma'_2} \leq \boldsymbol{\sigma_2}$*, and*
- *for the marking M'_1 reached after firing σ'_1 from M_0, i.e., $(N, M_0)[\sigma'_1\rangle(N, M'_1)$, it holds that $en_C(M'_1) \subseteq en_C(M) \cup [t]_C$.*

Proof. We proceed by induction on $|\sigma_2|$. If $|\sigma_2| = 0$, we can set $\sigma'_1 = \sigma_1$ and $\sigma'_2 = \sigma_2$ and the claim readily follows for $M'_1 = M_1$.

Otherwise, let $\delta = |en_C(M_1) \setminus (en_C(M) \cup [t]_C)|$. If $\delta = 0$, we can set $\sigma'_1 = \sigma_1$ and $\sigma'_2 = \sigma_2$ and the claim follows for $M'_1 = M_1$. So assume that $\delta > 0$. Then there exists some transition t_D which occurs in σ_2 for some cluster $D = [t_D]_C \in en_C(M_1) \setminus (en_C(M) \cup [t]_C)$ (since the cluster D is not enabled at M and cannot be disabled by firing t, and since $\delta > 0$ such cluster exists). Pick the first such transition t_D which occurs in σ_2, i.e., write $\sigma_2 = \tau_1 \langle t_D \rangle \tau_2$ where τ_1 only contains transitions t' such that $[t']_C \notin en_C(M_1)$ or $[t']_C \in en_C(M) \cup [t]_C$. Now, we can move t_D forward in front of t. Formally, let:

- $\sigma_1^{new} = \sigma_1 \langle t_D \rangle$,
- $\sigma_2^{new} = \tau_1 \tau_2$.

First, we observe that

$$\sigma^{new} = \sigma_1^{new} \langle t \rangle \sigma_2^{new} = (\sigma_1 \langle t_D \rangle) \langle t \rangle (\tau_1 \tau_2)$$

is a valid firing sequence from M_0 to M. Clearly, t_D is enabled at M_1, the marking reached after firing σ_1 from M_0, just by the choice of t_D. Moreover, firing t_D cannot disable t since $[t_D]_C \neq [t]_C$. Moreover, every transition $t' \in \langle \boldsymbol{\tau_1} \rangle$ also has the property that $[t']_C \neq [t_D]_C$. This is because $[t']_C \notin en_C(M_1)$ or $[t']_C \in en_C(M) \cup [t]_C$ while $[t_D]_C \in en_C(M_1) \setminus (en_C(M) \cup [t]_C)$. Hence, firing t_D cannot disable any transition in τ_1. Hence, $\sigma_1^{new} \langle t \rangle \tau_1$ is a valid firing sequence from M_0 to some marking M'. Since with the (permuted) orginal sequence $\sigma_1 \langle t \rangle \tau_1 \langle t_D \rangle$ we also reach M', it follows that σ^{new} is a valid firing sequence from M_0 to M.

We have $|\sigma_2^{new}| < |\sigma_2|$. Hence, by the induction hypothesis, we can rearrange $\sigma^{new} = \sigma_1^{new} \langle t \rangle \sigma_2^{new}$ into a firing sequence $\sigma' = \sigma'_1 \langle t \rangle \sigma'_2$ such that:

- $\boldsymbol{\sigma' = \sigma^{new} = \sigma}$,
- $\boldsymbol{\sigma'_2 \leq \sigma_2^{new} \leq \sigma_2}$, and
- for the marking M'_1 reached after firing σ'_1 from M_0, i.e., $(N, M_0)[\sigma'_1\rangle(N, M'_1)$, it holds that $en_C(M'_1) \subseteq en_C(M) \cup [t]_C$.

This concludes the proof (just pick σ'_1, σ'_2, and M'_1 to satisfy the claim). ☐

Theorem 10 (Shortest Sequences to Blocking Markings). *Let (N, M_0) with $N = (P, T, F)$ be an* LBFC*-system with a home cluster. Let $M \in [N, M_0\rangle$ be a reachable marking, let $t \in T$ be some transition, let $B \in [N, M_0\rangle$ be the unique blocking marking associated to the cluster $[t]_C$, and let $\sigma \in (T \setminus [t]_C)^*$ be a shortest firing sequence from M to B, i.e., $(N, M)[\sigma\rangle(N, B)$. Then, σ is 2-elementary. In particular, $|\sigma| \leq 2 \cdot |T|$.*

Proof. Assume that, to the contrary, σ was not elementary. Then σ can be written as:

$$\sigma = \eta\langle t_1\rangle\tau_1\langle t_2\rangle\tau_2\langle t_3\rangle\tau_3,$$

with $t_1, t_2, t_3 \in T$ from the same cluster, i.e., $D = [t_1]_C = [t_2]_C = [t_3]_C$, and where $\eta, \tau_1, \tau_2, \tau_3 \in (T\setminus[t]_C)^*$ are firing sequences not visiting the cluster $[t]_C$. Let M_1 be the marking reached after firing η from M, i.e., $(N, M)[\eta\rangle(N, M_1)$, let M_2 be the marking reached after firing $\langle t_1\rangle\tau_1$ from M_1, i.e., $(N, M_1)[\langle t_1\rangle\tau_1\rangle(N, M_2)$, and let M_3 be the marking reached after firing $\langle t_2\rangle\tau_2$ from M_2, i.e., $(N, M_2)[\langle t_2\rangle\tau_2\rangle(N, M_3)$. Of course, $(N, M_3)[\langle t_3\rangle\tau_3\rangle(N, B)$. In summary, we have the following situation:

$$M \xrightarrow{\eta} M_1 \xrightarrow{\langle t_1\rangle\tau_1} M_2 \xrightarrow{\langle t_2\rangle\tau_2} M_3 \xrightarrow{\langle t_3\rangle\tau_3} B.$$

We now expedite transitions starting at M_3 using Lemma 6. More precisely, we apply Lemma 6 to last part of the sequence σ (where we fix t_3 as t in the lemma):

$$M_2 \xrightarrow{\langle t_2\rangle\tau_2} M_3 \xrightarrow{\langle t_3\rangle\tau_3} B$$

This yields a rearranged firing sequence from M_2 to B of the form:

$$M_2 \xrightarrow{\langle t_2\rangle\tau'_2} M'_3 \xrightarrow{\langle t_3\rangle\tau'_3} B,$$

such that $en_C(M'_3) \subseteq D \cup [t]_C$ (in particular, at most two clusters are enabled at M'_3). Of course, if $en_C(M'_3) = [t]_C$, then by Theorem 8 we must have $M'_3 = B$ and we could skip firing $\langle t_3\rangle\tau'_3$ contradicting the minimality of σ. Hence, $en_C(M'_3) = D \cup [t]_C$ (note also that $D \neq [t]_C$, because σ does not visit $[t]_C$ by assumption).

We now repeat the same process of applying Lemma 6 to the part between M_1 and M'_3, i.e., to the subsequence:

$$M_1 \xrightarrow{\langle t_1\rangle\tau_1} M_2 \xrightarrow{\langle t_2\rangle\tau'_2} M'_3.$$

We end up with a rearranged firing sequence from M_1 to M_3' of the form:

$$M_1 \xrightarrow{\langle t_1\rangle\tau_1'} M_2' \xrightarrow{\langle t_2\rangle\tau_2''} M_3',$$

such that $en_C(M_2') \subseteq D \cup en_C(M_3') = D \cup [t]_C$. Again, if $en_C(M_2') = en_C(M_3')$, then $M_2' = M_3'$ by Theorem 8 and we could skip firing $\langle t_2\rangle\tau_2''$ contradicting the minimality of σ. Hence, $en_C(M_2') = D$.

For one last time, we apply Lemma 6 to the part between M and M_2', i.e., to the subsequence:

$$M \xrightarrow{\eta} M_1 \xrightarrow{\langle t_1\rangle\tau_1'} M_2',$$

to obtain a rearranged firing sequence from M to M_2' of the form:

$$M \xrightarrow{\eta'} M_1' \xrightarrow{\langle t_1\rangle\tau_1''} M_2',$$

such that $en_C(M_1') \subseteq D \cup en_C(M_2') = D$. Since $en_C(M_1') \neq \emptyset$, we have $en_C(M_1') = D$. Hence, by Theorem 8 we have $M_1' = M_2'$ and we could skip firing $\langle t_1\rangle\tau_1''$ contradicting the minimality of σ.

It turns out that our proof generalizes from blocking markings to arbitrary reachable markings. In general, we can show that in shortest sequencens, we can visit each cluster at most $k+1$ times where k is the number of clusters enabled at the target marking. This yields a quadratic bound on the length of shortest sequences between arbitrary pairs of reachable markings.

Theorem 11 (Shortest Sequences are $k{+}1$-elementary). *Let (N, M_0) with $N = (P, T, F)$ be an* LBFC*-system with home cluster. Let $M \in [N, M_0\rangle$ be a reachable marking with k enabled clusters, i.e., $|en_C(M)| = k$. Moreover, let $\sigma \in T^*$ be a shortest firing sequence from M_0 to M, i.e. $(N, M_0)[\sigma\rangle(N, M)$, and there is no subsequence $\eta \in T^*$, $\boldsymbol{\eta} < \boldsymbol{\sigma}$, satisfying $(N, M_0)[\eta\rangle(N, M)$. Then σ is $(k+1)$-elementary.*

Proof. For sake of illustration, let us first consider the case $k = 1$ (this is pretty much analgous to the proof of Theorem 10, except for that the sequence σ might now contain a transition from the cluster which is enabled at the target marking M). So let us assume that for some cluster D the shortest sequence σ from M_0 to M fires more than two transitions from D. Then, write σ as:

$$\sigma = \sigma_0\langle t_1\rangle\sigma_1\langle t_2\rangle\sigma_2\langle t_3\rangle\sigma_3,$$

for some transitions $t_1, t_2, t_3 \in D$ and firing sequences $\sigma_1, \sigma_2, \sigma_3 \in T^*$. First, we use Lemma 6 to rearrange $\sigma_2\langle t_3\rangle\sigma_3$ in such a way that after firing $\sigma_0\langle t_1\rangle\sigma_1\langle t_2\rangle\sigma_2$ we reach a marking M_3 which satisfies $en_C(M_3) \subseteq [D]_C \cup en_C(M)$ (note that since $k = 1$, only one cluster different from D can be enabled at M_3). We now apply Lemma 6 again to rearrange $\sigma_1\langle t_2\rangle\sigma_2$ in such a way that after firing $\sigma_0\langle t_1\rangle\sigma_1$ we reach a marking M_2 which satisfies $en_C(M_2) \subseteq [D]_C \cup en_C(M_3) = en_C(M_3)$. Now, if $en_C(M_2) = en_C(M_3)$, then $M_2 = M_3$ by Theorem 8 and we could skip firing $\langle t_2\rangle\sigma_2$ contradicting the minimality of σ. Hence, $en_C(M_2) \subset en_C(M_3)$. We repeat the same process one last time to rearrange $\sigma_0\langle t_1\rangle\sigma_1$ in such a way that after firing σ_0 we reach a marking M_1 which satisfies $en_C(M_1) \subseteq [D]_C \cup en_C(M_2) = en_C(M_2)$. Again, if $en_C(M_1) = en_C(M_2)$, then $M_1 = M_2$ by Theorem 8 and we could skip firing $\langle t_1\rangle\sigma_1$ contradicting the minimality of σ. Hence, $en_C(M_1) \subset en_C(M_2)$.

However, since $en_C(M_1) \subset en_C(M_2) \subset en_C(M_3) \subseteq [D]_C \cup en_C(M)$ and $|en_C(M)| = 1$, we have reached a contradiction and thus σ must be 2-elementary.

For the general case where $k > 1$ we proceed analogously. Assume that for some cluster D a shortest firing sequence σ from M_0 to M fires more than $k+1$ transitions from D. Then, we write σ as:

$$\sigma = \sigma_0\langle t_1\rangle\sigma_1\langle t_2\rangle\sigma_2 \cdots \langle t_{k+1}\rangle\sigma_{k+1}\langle t_{k+2}\rangle\sigma_{k+2},$$

for transitions $t_1, t_2, \ldots, t_{k+2} \in D$ and firing sequences $\sigma_1, \sigma_2, \ldots, \sigma_{k+2} \in T^*$. We now use Lemma 6 to rearrange $\sigma_{k+1}\langle t_{k+2}\rangle\sigma_{k+2}$ in such a way that after firing $\sigma_0\langle t_1\rangle\sigma_1\langle t_2\rangle\sigma_2 \cdots \langle t_k\rangle\sigma_k\langle t_{k+1}\rangle\sigma_{k+1}$ we reach a marking M_{k+2} with $en_C(M_{k+2}) \subseteq [D]_C \cup en_C(M)$. We proceed by rearranging $\sigma_k\langle t_{k+1}\rangle\sigma_{k+1}$ in such a way that after firing $\sigma_0\langle t_1\rangle\sigma_1\langle t_2\rangle\sigma_2 \cdots \langle t_{k-1}\rangle\sigma_{k-1}\langle t_k\rangle\sigma_k$ we reach a marking M_{k+1} which satisfies $en_C(M_{k+1}) \subseteq [D]_C \cup en_C(M_{k+2}) = en_C(M_{k+2})$. As argued before, if $en_C(M_{k+1}) = en_C(M_{k+2})$, then $M_{k+1} = M_{k+2}$ by Theorem 8 and we could skip firing $\langle t_{k+1}\rangle\sigma_{k+1}$ contradicting the minimality of σ. Hence, $en_C(M_{k+1}) \subset en_C(M_{k+2})$. We repeat the same process another step to rearrange $\sigma_{k-1}\langle t_k\rangle\sigma_k$ in such a way that after firing $\sigma_0\langle t_1\rangle\sigma_1\langle t_2\rangle\sigma_2 \cdots \langle t_{k-2}\rangle\sigma_{k-2}\langle t_{k-1}\rangle\sigma_{k-1}$ we reach a marking M_k which satisfies $en_C(M_k) \subseteq [D]_C \cup en_C(M_{k+1}) = en_C(M_{k+1})$. Again, if $en_C(M_k) = en_C(M_{k+1})$, then $M_k = M_{k+1}$ by Theorem 8 and we could skip firing $\langle t_k\rangle\sigma_k$ contradicting the minimality of σ. Hence, $en_C(M_k) \subset en_C(M_{k+1})$.

By repeating the same line of argument, we obtain a sequence of markings $M_1, M_2, \ldots, M_{k+2}$ such that:

$$en_C(M_1) \subset en_C(M_2) \subset \cdots \subset en_C(M_{k+2}) \subseteq [D]_C \cup en_C(M).$$

Since $|en_C(M)| = k$, we have reached a contradiction and thus σ must be $(k+1)$-elementary.

An immediate consequence of the previous theorem is a quadratic upper bound on the length of shortest firing sequences in LBFC-systems with home cluster.

Corollary 3 (Shortest Sequences in LBFC-systems with Home Cluster). *Let (N, M_0) with $N = (P, T, F)$ be an LBFC-system with home cluster. Let $M \in [N, M_0\rangle$ be a reachable marking. Moreover, let $\sigma \in T^*$ be a shortest firing sequence from M_0 to M. Then $|\sigma| \leq |[N]_C| \cdot |T| \leq |T|^2$.*

Let us remark that our arguments in this section heavily relied on Theorem 8 which, in turn, only holds for LBFC-systems with a *home cluster*.

6 Conclusion

This year not only marks the 100th anniversary of Carl Adam Petri, but also the 31st anniversary of the Shortest Sequence Theorem by Desel and Esparza [12, 13] providing a cubic bound as best known upper bound for shortest firing sequences in LBFC-systems.

In this paper, we have proved that this cubic bound can be improved to a quadratic one in the presence of a home cluster (Theorem 11). Moreover, in one of our main results (Theorem 10), this cubic bound can be further improved to a linear one for reaching blocking markings. Our results shed new light on the structure of LBFC-systems with home cluster that can prove useful for algorithmic analysis techniques. For example, in algorithmic contexts where we need to construct firing sequences instead of only deciding reachability, our results provide much better worst-case guarantees. Along the way, we obtained several results of independent interest that might prove useful in future research on free-choice systems. For example, we showed that from every marking, the home cluster can always be reached via an *elementary firing sequence*, i.e., a firing sequence that contains at most one transition from every cluster (Theorem 6). A consequence is that an arbitrary set of enabled clusters can always be disabled via an elementary sequence as well (Theorem 9). To mention just one immediate application, this shows that the edit distance between a trace and a sound free-choice workflow net is at most linear (in the length of the trace and the number of transitions of the net). This upper bound can be used in order to improve heuristics for conformance checking in process mining, for example, for alignment computations [9].

References

[1] W. M. P. van der Aalst. "Verification of workflow nets." In: *Application and Theory of Petri Nets 1997*. ICATPN 1997. Ed. by P. Azéma and G. Balbo. Vol. 1248. LNCS. Berlin, Heidelberg: Springer Berlin Heidelberg, 1997, pp. 407–426. ISBN: 978-3-540-69187-7. DOI: 10.1007/3-540-63139-9_48.

[2] W. M. P. van der Aalst. "Markings in Perpetual Free-Choice Nets Are Fully Characterized by Their Enabled Transitions." In: *Application and Theory of Petri Nets and Concurrency*. Petri Nets 2018. Ed. by V. Khomenko and O. H. Roux. Vol. 10877. LNCS. Cham: Springer International Publishing, 2018, pp. 315–336. ISBN: 978-3-319-91268-4. DOI: 10.1007/978-3-319-91268-4_16.

[3] W. M. P. van der Aalst. "Lucent Process Models and Translucent Event Logs." In: *Fundamenta Informaticae* 169.1-2 (2019), pp. 151–177. DOI: 10.3233/FI-2019-1842.

[4] W. M. P. van der Aalst. "Free-choice Nets with Home Clusters are Lucent." In: *Fundamenta Informaticae* 181.4 (2021), pp. 273–302. DOI: 10.3233/FI-2021-2059.

[5] W. M. P. van der Aalst. "Reduction Using Induced Subnets to Systematically Prove Properties for Free-Choice Nets." In: *Application and Theory of Petri Nets and Concurrency*. Petri Nets 2021. Ed. by D. Buchs and J. Carmona. Vol. 12734. LNCS. Cham: Springer International Publishing, 2021, pp. 208–229. ISBN: 978-3-030-76983-3. DOI: `10.1007/978-3-030-76983-3_11`.

[6] W. v. d. Aalst. *Process Mining. Data Science in Action*. 2nd ed. 2016. SpringerLink : Bücher. Berlin ; Heidelberg ; New York ; Dordrecht ; London: Springer, 2016. ISBN: 978-3-662-49851-4. DOI: `10.1007/978-3-662-49851-4`.

[7] E. Best, J. Desel, and J. Esparza. "Traps Characterize Home States in Free Choice Systems." In: *Theor. Comput. Sci.* 101.2 (1992), pp. 161–176.

[8] E. Best and R. Devillers. *Petri Net Primer. A Compendium on the Core Model, Analysis, and Synthesis*. Cham: Springer Nature Switzerland, 2024. ISBN: 978-3-031-48278-6. DOI: `10.1007/978-3-031-48278-6`.

[9] J. Carmona, B. F. van Dongen, and M. Weidlich. "Conformance Checking: Foundations, Milestones and Challenges." In: *Process Mining Handbook*. Ed. by W. M. P. van der Aalst and J. Carmona. Vol. 448. LNBIP. Cham: Springer International Publishing, 2022. Chap. 5, pp. 155–190. ISBN: 978-3-031-08847-6. DOI: `10.1007/978-3-031-08848-3_5`.

[10] F. G. Commoner, A. W. Holt, S. Even, and A. Pnueli. "Marked Directed Graphs." In: *Journal of Computer and System Sciences* 5.5 (1971), pp. 511–523. DOI: `10.1016/S0022-0000(71)80013-2`.

[11] J. Desel and J. Esparza. "Reachability in cyclic extended free-choice systems." In: *Theoretical Computer Science* 114.1 (1993), pp. 93–118. DOI: `10.1016/0304-3975(93)90154-L`.

[12] J. Desel and J. Esparza. *Free Choice Petri Nets*. Cambridge Tracts in Theoretical Computer Science 40. Cambridge: Cambridge University Press, 1995. ISBN: 978-0-521-01945-3. DOI: `10.1017/CBO9780511526558`.

[13] J. Desel and J. Esparza. "Shortest Paths in Reachability Graphs." In: *Journal of Computer and System Sciences* 51.2 (1995), pp. 314–323. DOI: `10.1006/jcss.1995.1070`.

[14] J. Esparza. "Reduction and Synthesis of Live and Bounded Free Choice Petri Nets." In: *Inf. Comput.* 114.1 (1994), pp. 50–87.

[15] J. Esparza and M. Nielsen. "Decidability Issues for Petri Nets - a survey." In: *Bull. EATCS* 52 (1994), pp. 244–262.

[16] J. Esparza and M. Nielsen. "Decidability Issues for Petri Nets - a survey." In: *CoRR* abs/2411.01592 (2024).

[17] J. Esparza and M. S. Suárez. "Top-down synthesis of live and bounded free choice nets." In: *Applications and Theory of Petri Nets*. Vol. 524. Lecture Notes in Computer Science. Springer, 1990, pp. 118–139.

[18] C. Favre, D. Fahland, and H. Völzer. "The relationship between workflow graphs and free-choice workflow nets." In: *Information Systems* 47 (2015), pp. 197–219. DOI: `https://doi.org/10.1016/j.is.2013.12.004`.

[19] B. Gaujal, S. Haar, and J. Mairesse. “Blocking a transition in a free choice net and what it tells about its throughput.” In: *Journal of Computer and System Sciences* 66.3 (2003), pp. 515–548. DOI: 10.1016/S0022-0000(03)00039-4.
[20] P. J. Meyer, J. Esparza, and H. Völzer. “Computing the concurrency threshold of sound free-choice workflow nets.” In: *International Conference on Tools and Algorithms for the Construction and Analysis of Systems*. Springer. 2018, pp. 3–19.
[21] C. T. Schwanen, W. Pakusa, and W. M. P. van der Aalst. “Process Tree Alignments.” In: *Enterprise Design, Operations, and Computing*. EDOC 2024. Ed. by J. Borbinha, T. Prince Sales, M. Mira Da Silva, H. A. Proper, and M. Schnellmann. Vol. 15409. LNCS. Cham: Springer International Publishing, 2025. ISBN: 978-3-031-78337-1. DOI: 10.1007/978-3-031-78338-8_16.
[22] J. Wehler. “Simplified proof of the blocking theorem for free-choice Petri nets.” In: *Journal of Computer and System Sciences* 76.7 (2010), pp. 532–537. DOI: 10.1016/j.jcss.2009.10.001.